# Asymptotic behavior of solutions to a model of spiral crystal growth (Nonlinear Diffusive Systems and Related Topics)



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# Asymptotic behavior of solutions to a model of spiral crystal growth

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## 1 Introduction

We consider the following semilinear parabolic equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{div}(A(x)\nabla u) + f(x,u), & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.1}$$

Here, $\nu$ is the outer normal unit vector of $\partial\Omega$, $A(x)$ is a smooth positive function on $\overline{\Omega}$ and $f(x,u)$ is a smooth function that is $2\pi$-periodic in $u$.

Problem (1.1) is related to a model of spiral crystal growth. Spiral ledges have been observed on the surface of many kinds of crystals such as silicon carbide (SiC), calcogen, paraffin and polyethylene ([18]). Frank [4] originally proposed the screw dislocation mechanism for crystal growth. Screw dislocation is a kind of lattice defect and produces a line step on the crystal surface. The step provides a preferred site for atoms to bond and moves normal to itself as the atoms attach to it. Since the velocity of the line step is assumed to be the same at any point, the angular velocity at the center is larger than that at the edge. Thus, the dislocation proceeds in a spiral shape.

Kobayashi [8] has proposed the following reaction-diffusion equation as a model of the motion of screw dislocations:

$$\begin{cases} \tau\dfrac{\partial u}{\partial t} = \varepsilon^2\Delta u - \sin(u - \Theta(x)) + \gamma, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $\tau, \varepsilon > 0$ are small parameters, and $\gamma$ is a constant. Here the domain $\Omega \subset \mathbb{R}^2$ is defined by

$$\Omega = \widetilde{\Omega} \setminus \bigcup_{j=1}^{N} D_{r_j}(\xi_j),$$

where $\widetilde{\Omega}$ is a simply connected bounded domain in $\mathbb{R}^2$ and $D_{r_j}(\xi_j) \subset \widetilde{\Omega}$ is a closed disk with radius $r_j$ centered at $\xi_j$ for $j = 1, \ldots, N$. The function $\Theta(x)$ is defined by

$$\Theta(x) = \sum_{j=1}^{N} m_j \theta_j(x),$$

where $m_j \in \mathbb{Z}$ and $\theta_j(x)$ is the angle between $x - \xi_j$ and the $x_1$ axis.

Equation (1.2) has a gradient structure

$$\tau \frac{\partial u}{\partial t} = -\frac{\delta H}{\delta u}$$

with the "free energy" functional $H$ defined by

$$H = \int_{\Omega} \left\{ \frac{\varepsilon^2}{2} |\nabla u|^2 - \cos(u - \Theta(x)) - \gamma u \right\} dx.$$

Here the unknown function $u(x,t)$ represents the local height of the crystal surface and is normalized in order that $2\pi$ denotes the size of a unit molecule. In this model, we assume that there are $N$ dislocations on the surface with fixed core regions $D_{r_j}(\xi_j)$ $(j = 1, \ldots, N)$ and that the initial height is given approximately by $\Theta(x)$. Actually, spiral growth with a hollow core at the center can be observed on the surface of SiC crystal ([18]).

Our main interest is the long-time behavior of solutions of (1.1) (or (1.2)) which grow up as $t \to +\infty$. Some numerical experiments imply that equation (1.2) has a growing solution with time-periodic profile. The purpose of this paper is to show the existence, monotonicity and stability of such a solution. More precisely, as we will see later, equation (1.1) or (1.2) has a solution which satisfies

$$U(x, t+T) = U(x,t) + 2\pi, \quad x \in \Omega,\ t > 0, \tag{1.3}$$

for some $T > 0$.

## 2 Main results

Throughout this paper, we assume that $A(x)$ and $f(x,u)$ are smooth functions satisfying the following conditions:

(A1) $A(x) > 0$ for all $x \in \overline{\Omega}$,

(A2) $f(x,u)$ is $2\pi$-periodic in $u$.

It is known that, for any $u_0 \in C(\overline{\Omega})$, a solution $u(x,t)$ of (1.1) with initial data $u(\cdot,0) = u_0$ exists globally in time, since $f$ is a bounded function (see [6], [12]). For $u_1$, $u_2 \in C(\overline{\Omega})$ we write

$$\begin{array}{lll} u_1 \le u_2 & \text{if} \;\; u_1(x) \le u_2(x) & \text{for all } x \in \overline{\Omega}, \\ u_1 < u_2 & \text{if} \;\; u_1(x) \le u_2(x) & \text{for all } x \in \overline{\Omega} \text{ and } u_1 \not\equiv u_2, \\ u_1 \ll u_2 & \text{if} \;\; u_1(x) < u_2(x) & \text{for all } x \in \overline{\Omega}. \end{array} \tag{2.1}$$

Let $\{S(t)\}_{t\ge 0}$ be the semiflow on $C(\overline{\Omega})$ generated by (1.1). In other words, the map $S(t)$ on $C(\overline{\Omega})$ is defined by $S(t)u_0 = u(\cdot,t)$ for each $t \ge 0$, where $u(x,t)$ is the solution of (1.1) with initial data $u(\cdot,0) = u_0$. The strong maximum principle ([17]) shows that $S(t)$ is strongly order-preserving ([10]), that is, $u_1 < u_2$ implies $S(t)u_1 \ll S(t)u_2$ for each $t > 0$. Further the standard parabolic estimate ([12]) shows that $S(t)$ is a compact map on $C(\overline{\Omega})$ for each $t > 0$. Since $f$ is $2\pi$-periodic in $u$, the semiflow $\{S(t)\}_{t\ge 0}$ also satisfies

$$S(t)(u_0 + 2k\pi) = S(t)u_0 + 2k\pi, \quad t \ge 0 \tag{2.2}$$

for all $u_0 \in C(\overline{\Omega})$ and $k \in \mathbb{Z}$.

In what follows, $\zeta(x,t)$ denotes the solution of (1.1) with initial data $\zeta(\cdot,0) \equiv 0$ and

$$\begin{aligned} \zeta^* &= \limsup_{t\to+\infty} \max_{x\in\overline{\Omega}} \zeta(x,t), \\ \zeta_* &= \liminf_{t\to+\infty} \min_{x\in\overline{\Omega}} \zeta(x,t). \end{aligned}$$

When both $\zeta^*$ and $\zeta_*$ are finite, the set $\{S(t)u_0 \mid t \ge 0\}$ is bounded in $C(\overline{\Omega})$ for any $u_0 \in C(\overline{\Omega})$. Since equation (1.1) has a Lyapunov functional, by virtue of the results of Matano [9], the $\omega$-limit set of $u_0$ is nonempty and is contained in the set of equilibria of (1.1).

Concerning the asymptotic behavior of growing-up solutions of (1.1), we obtain the following results:

**Theorem A** *Suppose that $\zeta^* = +\infty$.*

(i) *There exists a solution $U(x,t)$ of (1.1) and a positive constant $T$ such that*

$$U(x,t+T) = U(x,t) + 2\pi, \quad x \in \Omega, \ t \geq 0. \tag{2.3}$$

(ii) *The solution $U$ is stable in the sense of Lyapunov and is strictly monotone increasing in $t$, that is,*

$$U_t(x,t) > 0, \quad x \in \overline{\Omega}, \ t > 0. \tag{2.4}$$

(iii) *The solution $U$ is exponentially stable up to time shift, that is, there exists a positive constant $\mu$ such that for any $u_0 \in C(\overline{\Omega})$ the solution $u(x,t)$ of (1.1) with initial value $u_0$ satisfies*

$$\|u(\cdot,t) - U(\cdot,t+\tau_0)\|_{C(\overline{\Omega})} \leq M_0 e^{-\mu t}, \tag{2.5}$$

*for all $t \geq 0$, where $\tau_0 \in \mathbb{R}$ and $M_0 > 0$ are constants depending on $u_0$.*

**Remark 2.1** It immediately follows from the above theorem that if $\zeta_* = -\infty$ then there exists a solution $U(x,t)$ of (1.1) satisfying

$$U(x,t+T) = U(x,t) - 2\pi, \quad x \in \Omega, \ t \geq 0$$

for some $T > 0$.

**Remark 2.2** By (2.3), we see that the solution $U(x,t)$ is written in the form

$$U(x,t) = \phi(x,t) + \frac{2\pi}{T}t, \tag{2.6}$$

where $\phi$ is $T$-periodic in $t$. Namah and Roquejoffre [13] have been studied the existence and the stability of solutions of similar form to (2.6) (they call such solutions *periodic fronts*) for other parabolic equations. The methods to prove the existence of such solutions in [13] are based on the Leray-Schauder degree theory. In the present paper, we use the strongly order-preserving property and compactness of the semiflow $\{S(t)\}_{t\geq 0}$ instead.

## 3 Existence, Monotonicity and Stability

In this section, we show the existence, monotonicity and stability of a growing-up solution with time-periodic profile.

The following lemma yields that the oscillation of $\zeta(\cdot,t)$ in $\overline{\Omega}$ is uniformly bounded in $t$.

**Lemma 3.1** *There exists a positive constant $M$ independent of $t$ such that*

$$\max_{x\in\overline{\Omega}}\zeta(x,t)-\min_{x\in\overline{\Omega}}\zeta(x,t)\leq M$$

*for all $t\geq 0$.*

**Proof** Define

$$\alpha=\sup_{(x,u)\in\overline{\Omega}\times\mathbf{R}}|f(x,u)|$$

and

$$\eta(x,t)=\zeta(x,t)-\frac{1}{|\Omega|}\int_{\Omega}\zeta(x,t)dx,$$

where $|\Omega|$ denotes the volume of $\Omega$. Then $\eta$ satisfies

$$\begin{cases}\dfrac{\partial\eta}{\partial t}=L\eta+h(x,t), & x\in\Omega,\ t>0,\\ \dfrac{\partial\eta}{\partial\nu}=0, & x\in\partial\Omega,\ t>0,\end{cases}\tag{3.1}$$

where $L$ is the restriction of $\operatorname{div}(A(x)\nabla)$ on $X_0=\{u\in C(\overline{\Omega})\mid\int_{\Omega}u(x)dx=0\}$ and $h(x,t)$ is a bounded function defined by

$$h(x,t)=f(x,\zeta(x,t))-\frac{1}{|\Omega|}\int_{\Omega}f(x,\zeta(x,t))dx.$$

We note that $L$ generates an analytic semigroup $\{e^{tL}\}_{t\geq 0}$ on $X_0$ and that

$$\eta(\cdot,t)=\int_0^t e^{(t-s)L}h(\cdot,s)ds.$$

Let $\lambda_1 > 0$ be the least positive eigenvalue of $-\mathrm{div}(A(x)\nabla)$ in $C(\overline{\Omega})$ with homogeneous Neumann boundary conditions. Then there exist constants $M > 0$ and $\lambda \in (0,\lambda_1)$ such that $\|e^{tL}u\|_{C(\overline{\Omega})} \leq Me^{-\lambda t}\|u\|_{C(\overline{\Omega})}$ for all $t \geq 0$ and $u \in X_0$. Therefore, we have

$$\|\eta(\cdot,t)\|_{C(\overline{\Omega})} \leq \int_0^t Me^{-\lambda(t-s)}\|h(\cdot,s)\|_{C(\overline{\Omega})}ds \leq \frac{2M\alpha}{\lambda},$$

hence

$$\max_{x\in\overline{\Omega}}\zeta(x,t)-\min_{x\in\overline{\Omega}}\zeta(x,t)=\max_{x\in\overline{\Omega}}\eta(x,t)-\min_{x\in\overline{\Omega}}\eta(x,t)\leq\frac{4M\alpha}{\lambda}.$$

The lemma is proved. □

**Proof of Theorem A (i)** When $\zeta^* = +\infty$, there exists a sequence $0 < t_1 < t_2 < \cdots \to +\infty$ such that

$$\max_{x\in C(\overline{\Omega})}\zeta(x,t_j)\to+\infty.$$

By Lemma 3.1, we can take a positive integer $m_j$ such that

$$0\leq\zeta(x,t_j)-2m_j\pi\leq M+2\pi,\quad x\in\overline{\Omega}$$

for all $j \in \mathbb{N}$. We fix a positive constant $\delta$ and put

$$w_j=S(\delta)(\zeta(\cdot,t_j)-2m_j\pi)=\zeta(\cdot,t_j+\delta)-2m_j\pi.$$

Since the map $S(\delta)$ is compact, replacing $\{t_j\}$ by its subsequence if necessary, we have $\lim_{j\to\infty} w_j = \varphi$ for some $\varphi \in C(\overline{\Omega})$. We define

$$l(t)=\inf\{\tau\geq0\mid\zeta(\cdot,t)+2\pi\leq\zeta(\cdot,t+\tau)\}.$$

Since $\zeta^* = +\infty$, the function $l(t)$ is well-defined for each $t \geq 0$. By the comparison theorem, $l(t)$ is positive and is monotone decreasing in $t$. Put

$T = \lim_{t\to+\infty} l(t)$. Since $\zeta(\cdot,t)+2\pi \le \zeta(\cdot,t+l(t))$ for $t \ge 0$, letting $t = t_j+\delta$ and $j \to \infty$, we obtain $\varphi + 2\pi \le S(T)\varphi$. This implies $T > 0$.

Suppose that $\varphi + 2\pi < S(T)\varphi$. Then for any fixed $\rho > 0$, we have $S(\rho)(\varphi+2\pi) = S(\rho)\varphi + 2\pi \ll S(T+\rho)\varphi$. From this, for a sufficiently large $j_0 \in \mathbb{N}$, it follows that

$$S(\rho)w_{j_0} + 2\pi \ll S(T+\rho)w_{j_0}.$$

Therefore, there exists a small positive constant $\varepsilon \in (0,T)$ such that

$$S(\rho)w_{j_0} + 2\pi \le S(T-\varepsilon+\rho)w_{j_0},$$

and hence

$$\zeta(\cdot,t_{j_0}+\delta+\rho)+2\pi \le \zeta(\cdot,t_{j_0}+\delta+\rho+T-\varepsilon).$$

This implies $l(t_{j_0}+\delta+\rho) \le T-\varepsilon$, which contradicts the definition of $T$. Therefore $\varphi + 2\pi = S(T)\varphi$ holds and thus $U(\cdot,t) = S(t)\varphi$ satisfies (2.3). □

**Proof of Theorem A (ii)** Fix $t > 0$ and set

$$t_0 = \inf\{\tau > 0 \mid U(\cdot,t) \le U(\cdot,t+\tau)\} \le T.$$

Suppose that $t_0 > 0$. Then $U(\cdot,t) < U(\cdot,t+t_0)$ implies

$$U(\cdot,t)+2\pi = S(T)U(\cdot,t) \ll S(T)U(\cdot,t+t_0) = U(\cdot,t+t_0)+2\pi,$$

which contradicts the definition of $t_0$. Therefore $t_0 = 0$ and hence $U_t(\cdot,t) \ge 0$ holds. Furthermore, by the strong maximum principle we have (2.4).

Next we show that $U$ is stable in the sense of Lyapunov. For any $\varepsilon > 0$, take $\delta_0 > 0$ satisfying

$$\sup_{t\in[0,T]} \|U(\cdot,t+\delta_0) - U(\cdot,t-\delta_0)\|_{C(\overline{\Omega})} < \varepsilon$$

and set

$$\delta = \min\left\{\min_{x\in\overline{\Omega}}(U(x,\delta_0)-U(x,0)),\ \min_{x\in\overline{\Omega}}(U(x,0)-U(x,-\delta_0))\right\}.$$

By (2.4), the constant $\delta$ is positive. For any solution $u$ of (1.1) satisfying $\|u(\cdot,0) - U(\cdot,0)\|_{C(\overline{\Omega})} < \delta$, we have

$$U(\cdot,-\delta_0) < u(\cdot,0) < U(\cdot,\delta_0).$$

Therefore, by the positivity of $U_t$ and the comparison theorem, we obtain

$$U(\cdot,t-\delta_0) < U(\cdot,t) < U(\cdot,t+\delta_0),$$
$$U(\cdot,t-\delta_0) < u(\cdot,t) < U(\cdot,t+\delta_0),$$

hence

$$\|u(\cdot,t) - U(\cdot,t)\|_{C(\overline{\Omega})} < \|U(\cdot,t+\delta_0) - U(\cdot,t-\delta_0)\|_{C(\overline{\Omega})} < \varepsilon$$

for all $t \geq 0$.

# 4 Asymptotic Stability

In this section we study the asymptotic stability of the growing-up solution $U$. For the proof, the monotonicity of $U$ in $t$ plays a crucial role.

The following lemma is a modified version of Property (B2) in [2], where Xinfu Chen has studied, among other things, the asymptotic stability of traveling waves in one space dimensional evolution equations with nonlocal terms.

**Lemma 4.1** *There exists a positive constant $d$ such that for any supersolution $w^+(x,t)$ and any subsolution $w^-(x,t)$ of (1.1) satisfying $w^+(x,0) \geq w^-(x,0)$ for $x \in \overline{\Omega}$, we have*

$$w^+(x,1) - w^-(x,1) \geq d \int_\Omega \{w^+(y,0) - w^-(y,0)\}dy \tag{4.1}$$

*for all $x \in \overline{\Omega}$.*

This lemma follows from the positivity of the fundamental solution ([5], [7]) for the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(A(x)\nabla u) & \text{in} \quad \Omega \times [0,+\infty), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times [0,+\infty). \end{cases}$$

**Remark 4.2** The constant $d$ satisfies

$$0 < d \leq e^{-\beta}/|\Omega|,$$

where $\beta = \sup_{(x,u)\in\overline{\Omega}\times\mathbf{R}} |f_u(x,u)|$.

Let $U(x,t)$ be the solution of (1.1) obtained in Theorem A (i). We define positive constants $M$, $m$ and $\delta_*$ by

$$M = \max_{(x,t)\in\overline{\Omega}\times\mathbf{R}} U_t(x,t), \quad m = \min_{(x,t)\in\overline{\Omega}\times\mathbf{R}} U_t(x,t), \quad \delta_* = \frac{dm|\Omega|}{2M}.$$

By Remark 4.2, the constant $\delta_*$ satisfies $0 < \delta_* < 1/2$.

**Lemma 4.3** *Let $u(x,t)$ be a solution of (1.1) such that*

$$U(x,t_0+\tau_0) \leq u(x,t_0) \leq U(x,t_0+\tau_0+h_0), \quad x \in \overline{\Omega}$$

*for some $t_0 \geq 0$, $\tau_0 \in \mathbb{R}$ and $h_0 > 0$. Then, for any $t \geq t_0+1$ it holds that*

$$U(x,t+\tau_1) \leq u(x,t) \leq U(x,t+\tau_1+h_1), \quad x \in \overline{\Omega}, \tag{4.2}$$

*where $\tau_1 \in \{\tau_0, \tau_0+\delta_* h_0\}$ and $h_1 = (1-\delta_*)h_0$.*

**Proof** We may assume $t_0 = 0$ without loss of generality. By the comparison theorem,

$$U(x,t+\tau_0) \leq u(x,t) \leq U(x,t+\tau_0+h_0), \quad x \in \overline{\Omega}, \ t \geq 0. \tag{4.3}$$

Since

$$\int_\Omega \{U(y,\tau_0+h_0) - U(y,\tau_0)\}dy \geq m|\Omega|h_0,$$

either of the following holds:

(i) $\displaystyle\int_\Omega \{u(y,0) - U(y,\tau_0)\}dy \geq m|\Omega|h_0/2,$

(ii) $\displaystyle\int_\Omega \{U(y,\tau_0+h_0) - u(y,0)\}dy \geq m|\Omega|h_0/2.$

Here we consider only the case (i), since the other is treated similarly. By Lemma 4.1,

$$u(x,1) - U(x,1+\tau_0) \geq d\int_\Omega \{u(y,0) - U(y,\tau_0)\} \geq dm|\Omega|h_0/2$$

for $x \in \overline{\Omega}$. Since $U(x,1+\tau_0+\delta_* h_0) - U(x,1+\tau_0) \leq M\delta_* h_0 = dm|\Omega|h_0/2$, we have $u(x,1) \geq U(x,1+\tau_0+\delta_* h_0)$ for $x \in \overline{\Omega}$, hence

$$u(x,t) \geq U(x,t+\tau_0+\delta_* h_0), \quad x \in \overline{\Omega}, \ t \geq 1. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain the inequality (4.2) with $\tau_1 = \tau_0 + \delta_* h_0$ and $h_1 = \tau_0 + h_0 - \tau_1 = (1-\delta_*)h_0$. □

**Proof of Theorem A (iii)** Let $u_0 \in C(\overline{\Omega})$ and $u(x,t)$ be the solution of (1.1) with initial data $u_0$. We take $\tau_0 \in \mathbb{R}$ and $h_0 > 0$ satisfying

$$U(x,\tau_0) \leq u_0(x) \leq U(x,\tau_0+h_0), \quad x \in \overline{\Omega}.$$

It follows from Lemma 4.3 and a mathematical induction that for any $k \in \mathbb{N}$, $t \in [k,k+1)$ and $x \in \overline{\Omega}$,

$$U(x,t+\tau_k) \leq u(x,t) \leq U(x,t+\tau_k+h_k)$$

with $\tau_k \in \{\tau_{k-1}, \tau_{k-1}+\delta_* h_{k-1}\}$, $h_k = (1-\delta_*)h_{k-1}$. Therefore we obtain

$$U(x,t+\tau(t)) \leq u(x,t) \leq U(x,t+\tau(t)+h(t)), \quad x \in \overline{\Omega}, \ t \geq 0,$$

where $\tau(t) = \tau_{[t]}$, $h(t) = h_{[t]}$ and $[t]$ is the largest integer less than or equal to $t$. By the definition of $\tau(t)$ and $h(t)$,

$$\begin{aligned} &h(t) = (1-\delta_*)^{[t]}h_0, \\ &0 \leq \tau(t_1) - \tau(t_2) \leq \{(1-\delta_*)^{[t_2]} - (1-\delta_*)^{[t_1]}\}h_0, \end{aligned}$$

for any $t \geq 0$ and $t_1 > t_2 \geq 0$. Thus the limit $\lim_{t\to+\infty}\tau(t) = \tau_0$ exists and satisfies $0 \leq \tau_0 - \tau(t) \leq (1-\delta_*)^{[t]}h_0$. Hence, letting $\mu = -\log(1-\delta_*) > 0$, we have

$$\|u(\cdot,t) - U(\cdot,t+\tau_0)\|_{C(\overline{\Omega})} \leq M_0 e^{-\mu t}, \quad t \geq 0$$

with $M_0 = Mh_0/(1-\delta_*)$. □

# 5 Spiral Traveling Wave Solutions

In this section, we consider the special case where $\Omega$ is a 2-dimensional annulus $\Omega = \{x \in \mathbb{R}^2 \mid a < |x| < b\}$ and (1.2) is of the form

$$\begin{cases} u_t = \Delta u + f(u - \sigma\theta), & x \in \Omega,\ t > 0, \\ u_r = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{5.1}$$

Here $\sigma$ is a positive integer, $(r, \theta)$ denotes the polar coordinates of $x \in \overline{\Omega}$. We assume that $f$ is a smooth $2\pi$-periodic function satisfying

$$\int_0^{2\pi} f(u)du > 0. \tag{5.2}$$

Note that (5.1) is $G$-equivariant, where the action of the group $G = \{g_\alpha \mid \alpha \in \mathbb{R}\}$ is defined by

$$(g_\alpha u)(r, \theta) = u(r, \theta - \alpha) + \sigma\alpha.$$

By the condition (5.2), one can see that $\zeta^* = +\infty$, where $\zeta^*$ is defined in Section 2. Hence, the following corollary follows from Therorem A and the $G$-equivariance of (5.1). See [15] and [16] for details.

**Corollary B**

(i) *There exists a solution $U(x,t)$ of (5.1) which is written in the form*

$$U(x,t) = \phi(r, \theta - \omega t) + \sigma\omega t, \quad x \in \Omega, \ t > 0$$

*for some $\phi \in C(\overline{\Omega})$ and $\omega > 0$. Moreover, $\phi = \phi(r,\theta)$ is $2\pi/\sigma$-periodic in $\theta$.*

(ii) *The solution $U$ is stable in the sense of Lyapunov and is strictly monotone increasing in $t$. Furthermore, $U$ is exponentially stable up to time shift.*

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